Evolutionary games

Saul Mendoza-Palacios

Economic Studies Center of El Colegio de México

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Basic model

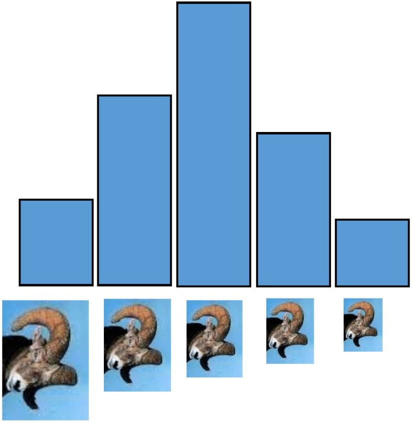
Let $I := \{1, 2, ..., n\}$ be the set of different species (or players). Each individual of the specie $i \in I$, can choose a single element in a set of characteristic (strategies or actions) $A_i := \{a_i^1, a_i^2, ..., a_i^{K_i}\}$. Let n_i^h be the number of individuals of species i that chose the action $a_i^h \in A^i$, then the total number of individuals of the specie i is

$$N_i = \sum_{h=1}^{K_i} n_i^h \quad \forall \quad h = 1, ..., K_i, \quad i \in I. \tag{1}$$

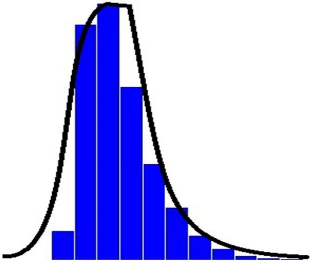
The proportion of the population of the species i who chose the action a_i^h is given by the fraction

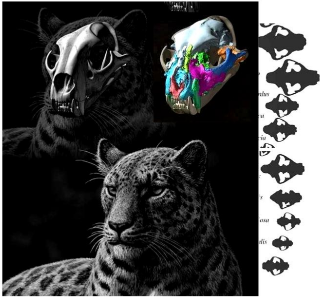
$$\mu_i(a_i^h) = \frac{n_i^h}{N_i} \geq 0 \quad \forall \quad h = 1, ..., K_i, \quad i \in I, \tag{2}$$

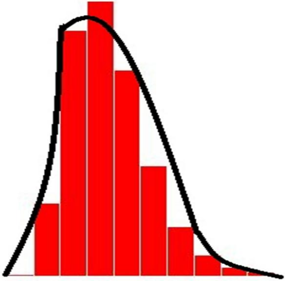
and the distribution of the population in relation their actions is describe by the vector $\mu_i = (\mu_i^1, \mu_i^2, ..., \mu_i^{K_i})$ where $\mu_i^h := \mu_i(a_i^h)$ and $\sum_{h=1}^{K_i} \mu_i^h = 1$.

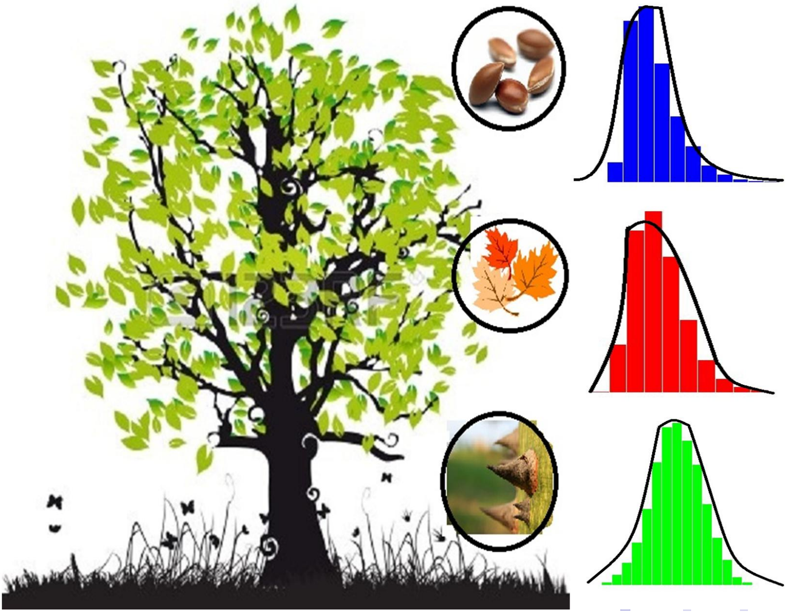



Pinzones de las Galápagos







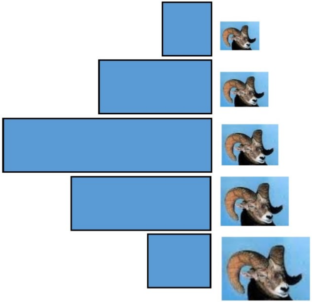


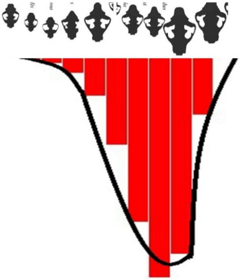
For each individual of the species i we assigned a payoff function $U_i : A \to R$ (where $A = A_1 \times ... \times A_n$) which explains its relationships with individuals of other species. The expected payoff of a individual of the species i who chose the action a_i^h and the other species have the population distributions $\mu_{-i} := (\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_n)$ is given by

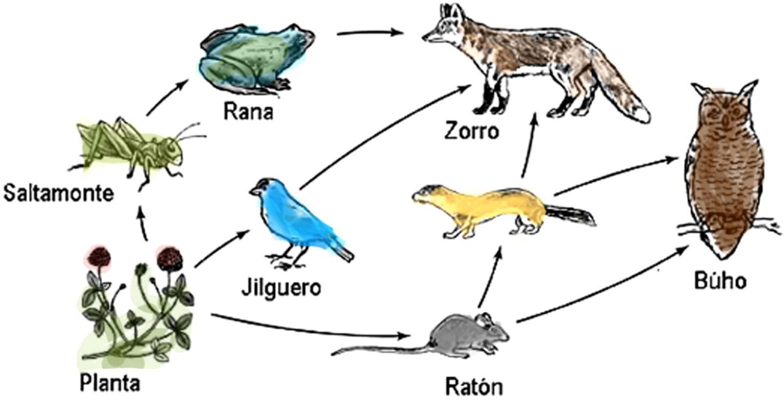
$$\begin{aligned} & J_i(a_i^h, \mu_{-i}) \\ = & \sum_{k=1}^{K_1} ... \sum_{s=1}^{K_{i-1}} \sum_{m=1}^{K_{i+1}} ... \sum_{q=1}^{K_n} \mu_1^k \cdots \mu_{i-1}^s \mu_{i+1}^m \cdots \mu_n^q U_i(a_1^k, ..., a_{i-1}^s, \mathbf{a}_i^h, a_{i+1}^m, ..., a_n^q) \end{aligned} \tag{3}$$

and the expected payoff of the species i, when its population distribution is μ_i is given by

$$J_i(\mu_i, \mu_{-i}) = \sum_{h=1}^{K_i} \mu_i^h J_i(a_i^h, \mu_{-i}) \tag{4}$$






Rana
Zorro
Saltamonte
Jilguero
Búho
Planta
Ratón

The replicator dynamics

Suppose that given a net birth rate γ_i for species i the dynamic of the of the subpopulation h is given by the following equation

$$\dot{n}_i^h(t) = [\gamma_i + J_i(a_i^h, \mu_{-i}(t))]n_i^h(t) \qquad \forall \quad h = 1, ..., K_i, \quad i \in I, \tag{5}$$

but we are interesting in the dynamic of the population distribution of each space. Since (by 2)

$$n_i^h = \mu_i^h N_i, \tag{6}$$

we have

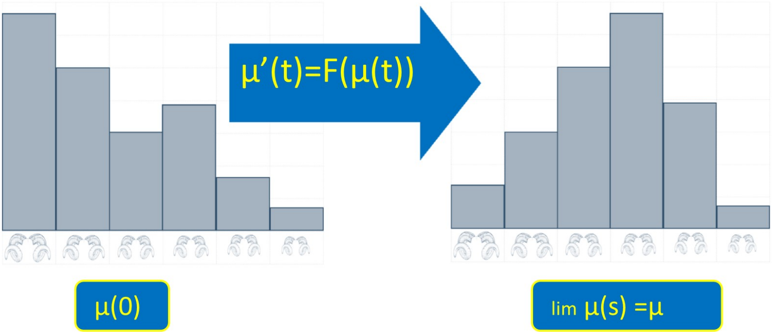
$$\dot{\mu}_i^h = \frac{1}{N_i}[\dot{n}_i^h - \mu_i^h \dot{N}_i] \qquad \forall \quad h = 1, ..., K_i, \quad i \in I, \tag{7}$$

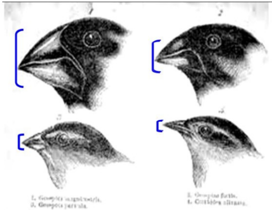
and also (by 1)

$$\dot{N}_i = \sum_{h=1}^{K_i} \dot{n}_i^h, \tag{8}$$

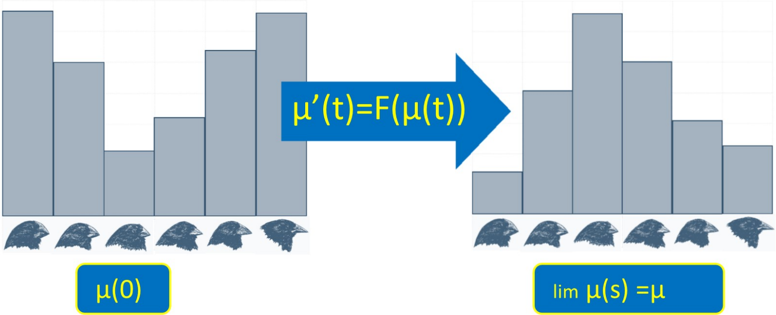
then we obtain the replicator dynamics

$$\dot{\mu}_i^h(t) = [J_i(a_i^h, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t))]\mu_i^h(t) \qquad \forall \quad h = 1, ..., K_i, \quad i = 1, ..., n. \tag{9}$$


μ'(t)=F(μ(t))
μ(0)
lim μ(s) =μ



Pinzones de las Galápagos



Asymmetric Evolutionary Games

We shall be working with a special class of asymmetric evolutionary games which can be described as

$$\left[I, \left\{ \mathbb{P}(A_i) \right\}_{i \in I}, \left\{ J_i(\cdot) \right\}_{i \in I}, \left\{ \dot{\mu}_i(t) = F_i(\mu(t)) \right\}_{i \in I} \right], \tag{10}$$

where

i) $I = \{1, ..., n\}$ is the finite set of players;

ii) for each player $i \in I$ we have a set of mixed actions $\mathbb{P}(A_i)$ and a payoff function $J_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{R}$; and

iii) the replicator dynamics $F_i(\mu(t))$, where

$$\dot{\mu}_i^h(t) = [J_i(a_i^h, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t))]\mu_i^h(t) \quad \forall \;\; h = 1, ..., K_i, \quad i = 1, ..., n. \tag{11}$$

Nash Equilibrium and SUP

Definition

Let Γ be a normal form game. A vector μ^* in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$ is called an equilibrium if, for all $i \in I$,

$$J_i(\mu_i^*, \mu_{-i}^*) \geq J_i(\mu_i, \mu_{-i}^*) \quad \forall \mu_i \in \mathbb{P}(A_i).$$

Definition

A vector $\mu^* \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times ... \times \mathbb{P}(A_n)$ is called a strong uninvadable profile (SUP) if the following holds: There exists $\epsilon > 0$ such that for any μ with $\|\mu - \mu^*\|_\infty < \epsilon$, and every $i \in I$, $J_i(\mu_i^*, \mu_{-i}) > J_i(\mu_i, \mu_{-i})$ *if* $\mu_i \neq \mu_i^*$.

Principal results

i) If $\mu^* = (\mu_1^*, ..., \mu_n^*)$ is a Nash equilibrium of Γ, then μ^* is a critical point of the replicator dynamics, i.e., $F_i(\mu^*) = 0$ for all $i \in I$.

ii) If μ^* be a SUP , then μ^* is an Nash equilibrium of Γ.

iii) If μ^* be a SUP , then μ^* is asymptotically stable point of the replicator dynamics.

iv) If μ^* is asymptotically stable point of the replicator dynamics, then it is a Nash equilibrium for Γ

Symmetric evolutionary games

We can obtain a symmetric evolutionary game when $I := \{1, 2\}$ and the sets of actions and payoff functions are the same for both players, i.e., $A = A_1 = A_2$ and $U(a, b) = U_1(a, b) = U_2(b, a)$, for all $a, b \in A$. As a consequence, the sets of mixed actions and the expected payoff functions are the same for both players , i.e., $\mathbb{P}(A) = \mathbb{P}(A_1) = \mathbb{P}(A_2)$ and $J(\mu, \nu) = J_1(\mu, \nu) = J_2(\nu, \mu)$, for all $\mu, \nu \in \mathbb{P}(A)$. This kind of model determines the dynamic interaction of strategies of a unique species through the replicator dynamics

$$\dot{\mu}^h(t) = [J(a^h, \mu(t)) - J(\mu(t), \mu(t))]\mu^h(t) \qquad \forall \quad h = 1, ..., m. \tag{12}$$

Finally, as in (10), we can describe a symmetric evolutionary games as

$$\left[I = \{1, 2\}, \quad \mathbb{P}(A), \quad J(\cdot), \quad \mu'(t) = F(\mu(t))\right]. \tag{13}$$

Evolutionary Game (EG)

$$\left[I = \{1,2\}, \quad \mathbb{P}(A), \quad J(\cdot), \quad \mu'(t) = F(\mu(t)) \right].$$

Normal Form Game (NFG)

$$[I = \{1,2\}, \quad \mathbb{P}(A), \quad J(\cdot),].$$

Definition

Let Γ_s be a symmetric normal form game. A vector μ^* in $\mathbb{P}(A)$ is called an Nash equilibrium strategy (NES) if (μ^*, μ^*) is a NE for Γ_s. That is

$$J(\mu^*, \mu^*) \geq J(\mu, \mu^*) \quad \forall \mu \in \mathbb{P}(A).$$

Definition

A probability measure $\mu^* \in \mathbb{P}(A)$ is called an strongly uninvadable strategy (SUS) if there exists $\epsilon > 0$ such that for any μ with $\|\mu - \mu^*\| < \epsilon$, it follows that $J(\mu^*, \mu) > J(\mu, \mu)$.

Principal results

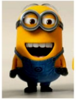
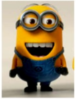
i) If μ^* is a NES of Γ_s, then μ^* is a critical point of the replicator dynamics, i.e., $F(\mu^*) = 0$.

ii) If μ^* be a SUS , then μ^* is an NES of Γ.

iii) If μ^* be a SUS, then μ^* is asymptotically stable point of the replicator dynamics.

iv) If μ^* is asymptotically stable point of the replicator dynamics, then it is a NES for Γ_s

Hawk-Dove game

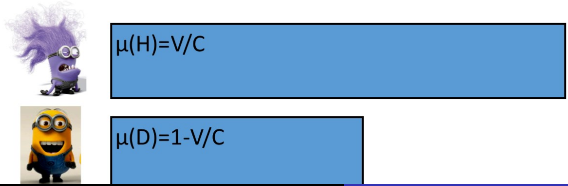
		
	$\frac{V-C}{2}, \quad \frac{V-C}{2}$	$V, \quad 0$
	$0, \quad V$	$\frac{V}{2}, \quad \frac{V}{2}$

For $0<V<C$ the symmetric Nash equilibrium is

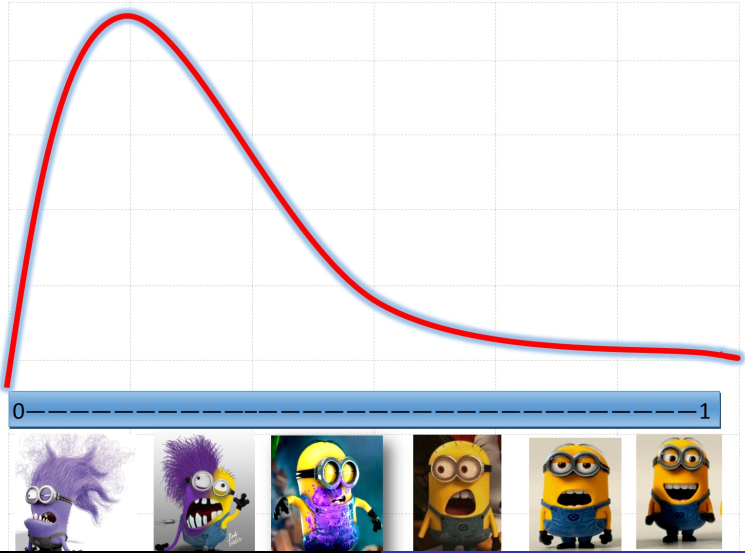
$$\mu(H)=V/C$$

and

$$\mu(D)=1-V/C$$



Graduated risk game



Graduated risk game

The graduated risk game is a symmetric game (proposed by Maynard Smith and Parker 1976), where two players compete for a resource of value $v > 0$. Each player selects the "level of aggression" for the game. This "level of aggression" is captured by a number $x \in [0, 1]$, where x is the probability that neither player is injured, and $\frac{1}{2}(1 - x)$ is the probability that player one (or player two) is injured. If the player is injured its payoff is $v - c$ (with $c > 0$), and hence the expected payoff for the player is

$$U(x, y) = \begin{cases} vy + \frac{v-c}{2}(1 - y) & y > x, \\ \frac{v-c}{2}(1 - x) & y \leq x, \end{cases}$$

where x and y are the "levels of aggression" selected by the player and her opponent, respectively.
If $v < c$, this game has the NES with the density function

$$\frac{d\mu^*(x)}{dx} = \frac{\alpha - 1}{2} x^{\frac{\alpha - 3}{2}}, \tag{14}$$

where $\alpha = \frac{c}{v}$.

References

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