Pontryagin's minimum principle applied to a discrete-time system with multiple modes

Webinar del Centro de Investigacion en Ciencias Físico Matemáticas

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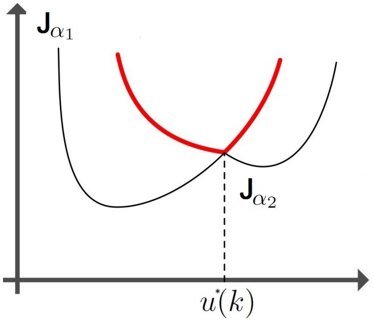
Contents

1. An LgQ problem with multiple modes

Take the n-dimensional linear dynamic system with multiple modes $\alpha \in \{\alpha_1, ..., \alpha_N\}$:

$$x_\alpha(k) = \underbrace{A_\alpha(k)x_\alpha(k-1) + B_\alpha(k-1)u(k-1) + d_\alpha(k)}_{f_\alpha(x_\alpha(k-1),u(k-1))}, \text{ for } k = 1, ..., T+1, x_\alpha(0) = x_0.$$

where $A_\alpha(k) \in \mathbb{R}^{n\times n}$, $B_\alpha(k) \in \mathbb{R}^{m\times n}$ are given control matrices, and $d_\alpha(k) \in \mathbb{R}^n$ is an exogenous vector of signals.



Let $S(k) \in \mathbb{R}^{n\times m}$ and define the generalized quadratic running-cost function:

$$g_k(x_\alpha, u) \;:=\; \frac{1}{2}\left[2x_\alpha^\top(k)S(k)u(k) + (u(k)-\bar{u}(k))^\top R(k)\,(u(k)-\bar{u}(k))\right],$$

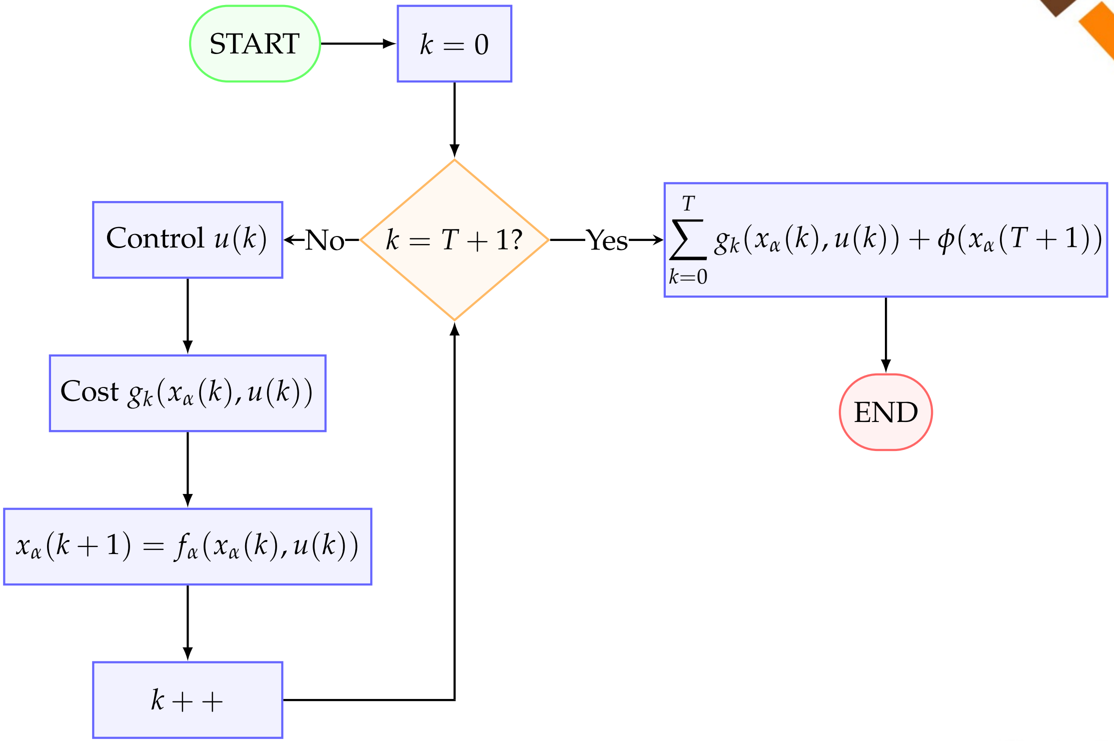
for $k = 1, ..., T$, as well as the terminal cost

$$\begin{aligned}\phi(x_\alpha(T+1)) \;:=\; & \frac{1}{2}\sum_{k=0}^{T}(x_\alpha(T+1)-\bar{x}(T+1))^\top Q(k)\,(x_\alpha(T+1)-\bar{x}(T+1)) \\ & + \frac{1}{2}(x_\alpha(T+1)-\bar{x}(T+1))^\top Q_f\,(x_\alpha(T+1)-\bar{x}(T+1)),\end{aligned}$$

where $\bar{x}$ is some given tracking signal, $\bar{u}$ is a desirable controller; and Q_f, Q and R are symmetric and semi-positive definite known matrices. And thus consider the generalized quadratic cost function which we intend to minimize:

$$h_\alpha(x_\alpha, u) := \sum_{k=0}^{T} g_k(x_\alpha(k), u(k)) + \phi(x_\alpha(T+1)). \tag{1.1}$$

With all these in mind, the situation is as follows...


START
$k = 0$
$k = T + 1?$
No
Yes
Control $u(k)$
Cost $g_k(x_\alpha(k), u(k))$
$x_\alpha(k+1) = f_\alpha(x_\alpha(k), u(k))$
$k++$
$\sum_{k=0}^{T} g_k(x_\alpha(k), u(k)) + \phi(x_\alpha(T+1))$
END

2. Un approccio classico/Une approche classique

The problem is to choose suitable open-loop controllers $u := (u(k) : k = 0, 1, ..., T)$ so as to minimize the worst-case scenario of the functional

$$\max \left[\sum_{k=0}^{T} g_k(x_\alpha(k), u(k)) + \phi(x_\alpha(T+1)) : \alpha \in \{\alpha_1, ..., \alpha_N\} \right], \tag{2.1}$$

subject to $x_\alpha(k+1) = f_\alpha(x_\alpha(k), u(k))$.

To this end, we start by fixing an arbitrary $\alpha \in \{\alpha_1, ..., \alpha_N\}$ and using a **classic technique**:

$$\begin{aligned} J_\alpha(x_\alpha, u, \psi_\alpha) := & \sum_{k=0}^{T} g_k(x_\alpha(k), u(k)) + \phi(x_\alpha(T+1)) \\ & + \sum_{k=0}^{T} \left[\psi_\alpha^\top (k+1) \left(f_\alpha(x_\alpha(k), u(k)) - x_\alpha(k+1) \right) \right]. \end{aligned}$$

$$
\begin{aligned}
J_\alpha(x_\alpha, u, \psi_\alpha) &= \sum_{k=0}^{T} \underbrace{\left[g_k(x_\alpha(k), u(k)) + \psi_\alpha^\top(k) f_\alpha(x_\alpha(k), u(k)) \right]}_{H_\alpha(x_\alpha(k), u(k), \psi_\alpha(k))} \\
&\quad - \sum_{k=0}^{T} \psi_\alpha^\top(k+1) x_\alpha(k+1) + \phi(x_\alpha(T+1)) \\
&= \sum_{k=0}^{T} \left[H_\alpha(x_\alpha(k), u(k), \psi_\alpha(k)) - \psi_\alpha^\top(k) x_\alpha(k) \right] \\
&\quad + \psi_\alpha^\top(0) x_\alpha(0) - \psi_\alpha^\top(T+1) x_\alpha(T+1) + \phi(x_\alpha(T+1)).
\end{aligned}
$$

Let ρ be an optimization scalar and re-write the original problem (2.1) as:

$$\text{Minimize } \rho \text{ subject to } J_\alpha(u) \le \rho \text{ for all } \alpha \in \{\alpha_1, ..., \alpha_N\}.$$

Which is equivalent to

$$\text{Minimize } \mathcal{L}(u, \rho, \lambda, \eta) := \eta\rho + \sum_{j=1}^{N} \lambda_j (J_{\alpha_j}(u) - \rho).$$

Now, by virtue of Kuhn-Tucker's Theorem of complementary slackness (see, for instance [2, Theorem 9.13]), $\eta \in \{0,1\}$, $\lambda \geq 0$, $(\eta, \lambda) \neq 0$, the vector (u^*, ρ^*) that solves this problem is such that

$$\sum_{j=1}^{N} \lambda_j \left(J_{\alpha_j} \left(x^*_{\alpha_j}, u^*, \psi_{\alpha_j} \right) - \rho^* \right) = 0, \quad (2.2)$$

$$\mathcal{L}\left(u^*, \rho^*, \lambda, \eta\right) = \eta \rho^*. \quad (2.3)$$

Now we present our first result, which gets the multiplier η out of the way, and gives us an important feature of the vector λ.

Proposition 2.1. *In the present context,* $\eta = 1$ *and* $\sum_{j=1}^{N} \lambda_j = 1$.

We take advantage of this result and introduce

$$S_N := \left\{ \lambda \in \mathbb{R}^{N\times 1} : \lambda_\alpha \geq 0 \text{ such that } \sum_{\alpha=1}^{N} \lambda_\alpha = 1 \right\}. \quad (2.4)$$

Proof of Proposition 2.1. Assume $\eta = 0$, then, since $J_\alpha(x_\alpha, u, \psi_\alpha) \leq \rho$ for all $\alpha \in \{\alpha_1, ..., \alpha_N\}$, we have that $\lambda > 0$, and thus, $\mathcal{L}(u, \rho, \lambda, \eta) \to -\infty$ as $\rho \to +\infty$, which yields that there are no values of (u^*, ρ^*) such that the minimum is attained. This is a contradiction. Then, necessarily, $\eta = 1$.

To see that $\sum_{j=1}^{N} \lambda_{\alpha_j} = 1$, we note that

$$\begin{aligned} \mathcal{L}(u, \rho, \lambda, 1) &= \rho + \sum_{j=1}^{N} \lambda_j J_{\alpha_j}\left(x_{\alpha_j}, u, \psi_{\alpha_j}\right) - \sum_{j=1}^{N} \lambda_j \rho \\ &= \rho \left(1 - \sum_{j=1}^{N} \lambda_j\right) + \sum_{j=1}^{N} \lambda_j J_{\alpha_j}\left(x_{\alpha_j}, u, \psi_{\alpha_j}\right). \end{aligned} \tag{2.5}$$

Due to (2.5), we can ensure that $\sum_{j=1}^{N} \lambda_j = 1$. Indeed:

Case 1. If $\sum_{j=1}^{N} \lambda_j > 1$, then the Lagrangian diverges. That is, $\mathcal{L}(u, \rho, \lambda, 1) \to -\infty$ as $\rho \to +\infty$, which yields that the minimum does not exist.

Case 2. If $\sum_{j=1}^{N} \lambda_j < 1$, then $\mathcal{L}(u, \rho, \lambda, 1) \to -\infty$ as $\rho \to -\infty$, which implies that the minimum does not exist either.

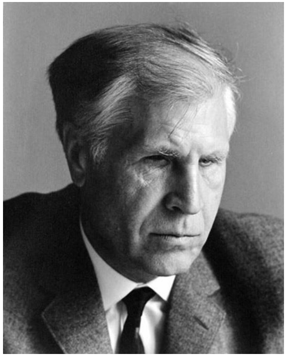
This completes the result. □

Proposition 2.1 allows us to use (2.2) and (2.3) to assert that the problem at hand is

$$\text{Minimize } \sum_{j=1}^{N} \lambda_j J_{\alpha_j}\left(x_{\alpha_j}, u, \psi_{\alpha_j}\right), \text{ with } \sum_{j=1}^{N} \lambda_j = 1. \tag{2.6}$$

3. An out-of-sight contribution in colors

Now we follow the approach of [3] to use L.S. Pontryagin's minimum principle (see [1]) in our problem.



- Consider the problem (2.6) with

$$J_{\alpha_j}\left(x_{\alpha_j}, u, \psi_{\alpha_j}\right) = \sum_{k=0}^{T}\left[H_{\alpha_j}\left(x_{\alpha_j}(k), u(k), \psi_{\alpha_j}(k)\right) - \psi_{\alpha_j}^{\top}(k) x_{\alpha_j}(k)\right]$$
$$+\psi_{\alpha_j}^{\top}(0) x_{\alpha_j}(0) - \psi_{\alpha_j}^{\top}(T+1) x_{\alpha_j}(T+1) + \phi(x_{\alpha_j}(T+1)),$$
$$H_{\alpha}\left(x_{\alpha}(k), u(k), \psi_{\alpha}(k)\right) = g_k(x_{\alpha}(k), u(k)) + \psi_{\alpha}^{\top}(k) f_{\alpha}(x_{\alpha}(k), u(k)),$$

and define the performance index

$$L(x, u, \lambda, \psi_{\alpha}) := \sum_{j=1}^{N} \lambda_j J_{\alpha_j}\left(x_{\alpha_j}, u, \psi_{\alpha_j}\right). \tag{3.1}$$

- Characterize (2.6) as:

$$\min_{u} \max_{\lambda \in \left\{\lambda \in \mathbb{R}^{N\times 1} : \lambda \geq 0, \sum_{j=1}^{N} \lambda_j = 1\right\}} L(x, u, \lambda, \psi) \text{ s.t. } u \in \mathbb{R}^{(T+1)\times 1}. \tag{3.2}$$

- Assume the existence of an optimal controller $u^* = (u^*(k) : k = 0, 1, ..., T)$, and let x^* be the associated trajectory.

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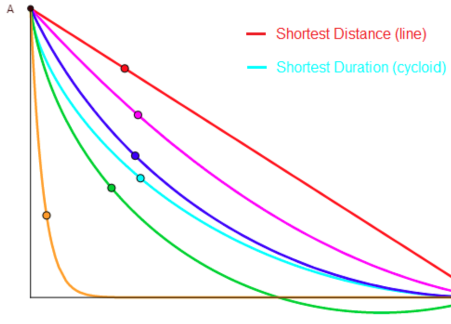
- Take $\varepsilon \in \mathbb{R}$ to define the perturbations of the controller and the trajectory at each stage:

$$
\begin{aligned}
u^{\varepsilon}(k) &:= u^{*}(k) + \varepsilon\beta_1(k) \text{ for } k = 0, 1, \ldots, T, \\
x^{\varepsilon}(k) &:= x^{*}(k) + \varepsilon\beta_2(k) \text{ for } k = 0, 1, \ldots, T+1,
\end{aligned}
$$

where $\beta_1(k)$ and $\beta_2(\ell)$ are pairwise linearly independent directions for $k, \ell = 0, \ldots, T+1$.



- Substitute u^ε and x^ε in (3.1) to obtain:

$$
\begin{aligned}
L\left(x^{\varepsilon}, u^{\epsilon}, \lambda, \psi_{\alpha}\right) &:= \sum_{j=1}^{N} \lambda_j J_{\alpha_j}\left(x^{\varepsilon}_{\alpha_j}, u^{\varepsilon}, \psi_{\alpha_j}\right) \\
= & \sum_{j=1}^{N} \lambda_j \left[\sum_{k=0}^{T}\left[H_{\alpha_j}\left(x^{\varepsilon}_{\alpha_j}(k), u^{\varepsilon}(k), \psi_{\alpha}(k)\right) - \psi^{\top}_{\alpha_j}(k) x^{\varepsilon}_{\alpha_j}(k)\right] + \psi^{\top}_{\alpha_j}(0) x^{\varepsilon}_{\alpha_j}(0)\right] \\
& - \sum_{j=1}^{N} \lambda_j \left[\psi^{\top}_{\alpha_j}(T+1) x^{\varepsilon}_{\alpha_j}(T+1) - \phi(x^{\varepsilon}_{\alpha_j}(T+1))\right].
\end{aligned}
$$

- Since u^* and x^* are optimal, we have that

$$
\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon} L\left(x^{\varepsilon}, u^{\epsilon}, \lambda, \psi_{\alpha}\right)\right|_{\varepsilon=0} = 0, \tag{3.3}
$$

$$
\left.\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} L\left(x^{\varepsilon}, u^{\epsilon}, \lambda, \psi_{\alpha}\right)\right|_{\varepsilon=0} \geq 0. \tag{3.4}
$$

Developing (3.3) yields:

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\sum_{j=1}^{N}\lambda_j\left[\sum_{k=0}^{T}\left[H_{\alpha_j}\left(x^{\varepsilon}_{\alpha_j}(k), u^{\varepsilon}(k), \psi_{\alpha_j}(k)\right) - \psi^{\top}_{\alpha_j}(k)x^{\varepsilon}_{\alpha_j}(k)\right] + \psi^{\top}_{\alpha_j}(0)x^{\varepsilon}_{\alpha_j}(0)\right] \\
&\quad -\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\sum_{j=1}^{N}\lambda_j\left[\psi^{\top}_{\alpha_j}(T+1)x^{\varepsilon}_{\alpha_j}(T+1) - \phi(x^{\varepsilon}_{\alpha_j}(T+1))\right] \\
&= \sum_{j=1}^{N}\sum_{k=0}^{T}\lambda_j\left(\frac{\partial}{\partial x}H_{\alpha_j}\left(x^{*}_{\alpha_j}(k), u^{*}(k), \psi_{\alpha_j}(k)\right) - \psi_{\alpha_j}(k)\right)^{\top}\beta_2(k) \\
&\quad + \sum_{j=1}^{N}\sum_{k=0}^{T}\lambda_j\left(\frac{\partial}{\partial u}H_{\alpha_j}\left(x^{*}_{\alpha_j}(k), u^{*}(k), \psi_{\alpha_j}(k)\right)\right)^{\top}\beta_1(k) \\
&\quad + \sum_{j=1}^{N}\lambda_j\psi^{\top}_{\alpha_j}(0)\beta_2(0) - \sum_{j=1}^{N}\lambda_j\left[\psi_{\alpha_j}(T+1) - \frac{\partial}{\partial x}\phi(x^{*}_{\alpha_j}(T+1))\right]^{\top}\beta_2(T+1).
\end{aligned}
$$

- Besides, by (3.4), $\frac{\partial^2}{\partial u^2}H_{\alpha_j}\left(x^{*}_{\alpha_j}(k), u^{*}(k), \psi_{\alpha_j}(k)\right) \le 0.$

We have thus proven this discrete-time *robust* version of Pontryagin-Boltyanskii-Gamkrelidze's principle.

Theorem 3.1. *Let* $u^* = (u^*(k) : k = 0, ..., T)$ *be an optimal control, and let* x^* *be the associated trajectory. Then, there exist adjoint vectors* $\psi_{\alpha_j}(k)$ *for* $k = 0, 1, ..., T+1$; $j = 1, ..., N$ *and* $\lambda \geq 0$ *with* $\sum_{j=1}^{N} \lambda_j \psi_{\alpha_j}(0) = 0$ *and* $\sum_{j=1}^{N} \lambda_j = 1$*, such that:*

$$\sum_{j=1}^{N} \lambda_j \left[\psi_{\alpha_j}(k) - \frac{\partial}{\partial x} H_{\alpha_j} \left(x^*_{\alpha_j}(k), u^*(k), \psi_{\alpha_j}(k) \right) \right] = 0, \tag{3.5}$$

$$\sum_{j=1}^{N} \lambda_j \left[\psi_{\alpha_j}(T+1) - \frac{\partial}{\partial x} \phi(x^*_{\alpha_j}(T+1)) \right] = 0, \tag{3.6}$$

$$\sum_{j=1}^{N} \lambda_j \left(\frac{\partial}{\partial u} H_{\alpha_j} \left(x^*_{\alpha_j}(k), u^*(k), \psi_{\alpha_j}(k) \right) \right)^{\top} = 0, \tag{3.7}$$

with $\lambda_j \left(J_{\alpha_j} \left(x^*_{\alpha_j}, u^*, \psi_{\alpha_j} \right) - \rho^* \right) = 0$ *for* $j = 1, ..., N$.

4. Robust optimal control for an affine LgQ problem

Use (3.5)-(3.7) found in Theorem 3.1 to write the minimax problem (2.6) as

$$\begin{aligned}
\mathbf{Q}_\lambda(k)\left(\mathbf{x}(k)-\bar{\mathbf{x}}(k)\right)+\mathbf{S}_\lambda^\top(k)u(k)+\mathbf{A}^\top(k)\Psi_\lambda(k+1)+\Psi_\lambda(k) &= 0,\\
\mathbf{R}_\lambda(k)\left(u(k)-\bar{u}(k)\right)+\frac{1}{2}\mathbf{S}_\lambda^\top(k)\mathbf{x}(k)+\mathbf{B}^\top(k)\Psi_\lambda(k+1) &= 0,\\
\mathbf{Q}_{\lambda,f}\left(\mathbf{x}(T+1)-\bar{\mathbf{x}}(T+1)\right)+\Psi_\lambda(T+1) &= 0,
\end{aligned}$$

where $\mathbf{A}(k) := \text{diag}\left[A_{\alpha_1}(k), ..., A_{\alpha_N}(k)\right] \in \mathbb{R}^{nN\times nN}$,
$\mathbf{B}(k) := \left[B_{\alpha_1}^\top(k), ..., B_{\alpha_N}^\top(k)\right]^\top \in \mathbb{R}^{mN\times nN}$, $\mathbf{Q}_{f,\lambda} := \text{diag}\left[\lambda_1 Q_f, ..., \lambda_N Q_f\right] \in \mathbb{R}^{nN\times nN}$,
$\mathbf{Q}_\lambda(k) := \text{diag}\left[\lambda_1 Q(k), ..., \lambda_N Q(k)\right] \in \mathbb{R}^{nN\times nN}$, $\mathbf{R}_\lambda(k) := \sum_{j=1}^N \lambda_j R(k)$,
$\mathbf{S}_\lambda(k) := \left[\lambda_1 S(k), ..., \lambda_N S(k)\right] \in \mathbb{R}^{n\times nm}$, $u(k), \bar{u}(k) \in \mathbb{R}^m$,
$\mathbf{x}(k) := \left[x_{\alpha_1}^\top(k), ..., x_{\alpha_N}^\top(k)\right] \in \mathbb{R}^{nN}$, $\bar{\mathbf{x}}(k) := \left[\bar{x}^\top(k), ..., \bar{x}^\top(k)\right] \in \mathbb{R}^{nN}$
and $\Psi_\lambda(k) := \left[\lambda_1 \psi_{\alpha_1}^\top(k), ..., \lambda_N \psi_{\alpha_N}^\top(k)\right]^\top \in \mathbb{R}^{nN}$.

Remark 4.1. *By virtue of the complementary slackness condition (2.2) referred to by Theorem 3.1,*

$$J_{\alpha_j}\left(x^*_{\alpha_j}, u^*, \psi_{\alpha_j}\right) = \rho^* \text{ for all } j = 1, ..., N.$$

This is a crucial fact for designing a numerical procedure that finds λ.

Remark 4.2. *The function $L(x, u, \lambda, \psi)$ depends only on Lagrange's multipliers λ. Indeed, by Remark 4.1, if we fix u^* and x^* in (3.5)-(3.7), then the function $L(x^*, u^*, \lambda, \psi)$ turns out to depend only on λ. Then,*

$$L(\lambda) := \max\{J_{\alpha_1}, ..., J_{\alpha_N}\}. \tag{4.1}$$

hence, in order to solve (3.2), we require now to solve a finite dimensional optimization problem that depends only on one *vector parameter:*

$$\lambda^* = \arg\min_{\lambda} L(\lambda). \tag{4.2}$$

Theorem 4.3. *If the matrix* $\mathbf{C}(k) := I + \mathbf{B}(k)(\mathbf{R}_\lambda(k))^{-1}(\mathbf{B}(k))^\top \mathbf{M}(k)$ *is non-singular, then* $u^*(k+1) = -\mathbf{P}_\lambda(k+1)\mathbf{x}(k) - \mathbf{p}_\lambda(k+1)$ *for* $k = 0, 1, ..., T+1$ *is optimal for* (3.2)*, where* $\mathbf{p}_\lambda(k) \in \mathbb{R}^{nN}$ *and* $\mathbf{P}_\lambda(k) \in \mathbb{R}^{nN \times nN}$ *is a symmetric matrix that solves:*

$$\mathbf{P}_\lambda(k+1) = (\mathbf{R}_\lambda(k+1))^{-1}\mathbf{B}(k+1)\mathbf{M}_\lambda(k+1)\Phi(k),$$

$$\mathbf{p}_\lambda(k+1) = \bar{u}(k+1)$$
$$+(\mathbf{R}_\lambda(k+1))^{-1}\mathbf{B}(k+1)\left[\mathbf{M}_\lambda(k+1)\gamma(k) + \mathbf{m}_\lambda(k+1) - \mathbf{Q}_\lambda(k+1)\bar{\mathbf{x}}(k+1)\right],$$

with $\mathbf{M}_\lambda(k+1) = \mathbf{Q}_\lambda(k) + (\mathbf{A}(k+1))^\top \mathbf{M}_\lambda(k+1)\left(\mathbf{C}(k+1)\right)^{-1}\mathbf{A}(k+1)$ *for* $k = 1, ..., T$*;* $\mathbf{m}_\lambda(k) = (\mathbf{A}(k+1))^\top\left(\mathbf{M}_\lambda(k+1)\Phi(k) + \mathbf{m}_\lambda(k+1) - \mathbf{Q}_\lambda(k+1)\bar{\mathbf{x}}(k+1)\right),$

$$\gamma(k) = (\mathbf{C}(k+1))^{-1}\mathbf{d}(k+1) + \mathbf{B}\left(k+1\right)\left(\mathbf{R}_\lambda\left(k+1\right)\right)^{-1}\left(\mathbf{B}(k+1)\right)^\top \bar{u}(k+1)$$
$$-\mathbf{B}\left(k+1\right)\left(\mathbf{R}_\lambda\left(k+1\right)\right)^{-1}\left(\mathbf{B}(k+1)\right)^\top\left(\mathbf{m}_\lambda(k+1) - \mathbf{Q}_\lambda(k+1)\bar{\mathbf{x}}(k+1)\right)$$

for $k = 0, 1, ..., T$*;* $\mathbf{m}_\lambda(T+1) = 0$ *and* $\mathbf{M}_\lambda(T+1) = Q_f$*. The weighting vector* λ^* *solves the finite-dimensional optimization problem* (4.2).

5. Value iteration for λ

We use a numerical procedure that extends [4, Theorem 21.15]. It projects consecutive iterations to the simplex S_N defined in (2.4). Assuming that $L(\lambda) > 0$ for all $\lambda \in S_N$ (if not, we can use an additive positive constant in (4.1)). To this end, define

$$\begin{aligned}
\lambda^{k+1} &= \pi_{S^N}\left\{\lambda^k + \frac{\gamma(k)}{\bar{L}\left(\lambda^k\right) + \varepsilon}\tilde{F}(\lambda^k)\right\}; \lambda^0 \in S_N, k = 0, ..., T, \\
\tilde{F}(\lambda^k) &= \left(\tilde{L}_{\alpha_1}, ..., \tilde{L}_{\alpha_N}\right), \\
\bar{L}(\lambda^k) &= \max_{\alpha \in \{\alpha_1, ..., \alpha_N\}} \tilde{L}_\alpha(\lambda^k),
\end{aligned}$$

where $\pi_{S_N}\{\cdot\}$ is the projection of an argument to the simplex S_N, and the new regularized functional $\tilde{L}_\alpha$ is defined as:

$$\tilde{L}_\alpha\left(\lambda\right) := \frac{\delta}{2}\left\|\lambda\right\|^2 + J_\alpha \text{ with } \delta \geq 0.$$

References

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Thank you for your attention!

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