Empirically Testing Predictions of an Attrition Warfare Model for the War in Ukraine

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Summary

The goal of this study is to empirically test hypotheses about wars of attrition by evaluating their predictions for the conflict in the Ukraine. Evaluation will occur after the war is over and authoritative data sources become available for analysis.

This pre-registration document presents two quantitative hypotheses that make opposite predictions about the course of the War in Ukraine: (1) the Economic Power hypothesis, which predicts a win for Ukraine and (2) the Casualties Rates hypothesis, which predicts a win for Russia. Additionally, I consider an alternative hypothesis, according to which the outcome will be determined by random unforeseen events.

The document includes four main parts:

1. An introduction providing the conceptual background and the rationale for this study.
2. The mathematical framework and a computational model that incorporates both Economic Power and Casualties Rates hypotheses as special cases.
3. An analysis plan that defines model outputs (what is predicted) and model inputs (parameter values and initial conditions), which need to be estimated from data.
4. An interim assessment (as of Summer 2023) using non-authoritative sources illustrating how, after the end of the war, input parameters will be estimated and the accuracy of predictions assessed.

At the time of pre-registration (November 2023) the conflict is still unresolved. Neither side has made significant territorial gains for over a year (since the late Fall of 2022). Furthermore, no authoritative source for data, needed to accurately estimate inputs, is currently available. Estimates published in the press differ wildly depending on the source. As a consequence, the alternative predictions discussed in the interim assessment should not be taken as predicting the future course of the conflict. They instead are meant to demonstrate how these specific scientific hypotheses about war dynamics will be assessed after the war concludes.

Notes

1. This work doesn't take partisan or ideological sides and doesn't consider the rights and wrongs of this war. Instead, the question that it addresses is, what is the dynamic of this conflict? And can its end be predicted on the basis of quantitative models? Approaching war in the spirit of scientific inquiry, evenhanded and dispassionate, is hard because war is such an ugly thing. Abolishing war should be one of the most important goals for humanity as a whole. But to do it effectively, we need to study it, and such is the goal of this study.
2. The central concept in this study is *scientific prediction*. The goal is not to predict the future, but to find out which of the hypotheses about war dynamics is supported by data. Currently (as of writing and publication of this article), the final outcome of the War in Ukraine is unknown, which makes prediction particularly challenging. But the goal is not to predict the future; rather, the goal is to determine which of the hypotheses performs best by quantitatively assessing the accuracy of their predictions.

3. I have considered using the OSF Registries portal for this study. However, the OSF pre-registration format is appropriate for a very different kind of science, for example, for experiments testing hypotheses in psychology. As a result, it requires answers to some questions that are irrelevant to this study, while not asking questions that are of key importance. This pre-registration document, therefore, is posted as a regular SocArxiv preprint.

Introduction

War is the most demanding of human undertakings and perhaps the most unpredictable process in human history (Turchin 2023: Chapter A1). Political leaders, pundits, and prophets are not shy about predicting the outcome of world conflicts, but their track record is not impressive. One type of conflicts, wars of attrition, may be more amenable to prediction because their outcomes are primarily determined by numerical quantities and their rates of change. There is a substantial literature on mathematical models of attritional warfare (Dupuy 1987, Turchin 2003). The goal of this study is to explore whether such models are capable of predicting war, with the ongoing War in Ukraine as a specific example.

The general perception of which side is winning in this conflict has fluctuated together with territory changing hands back and forth (Figure 1). But the fundamentals determining the overall trajectory of war stayed largely the same or changed slowly (more on this in section *The Mathematical Framework*). Two major predictions appealing to such fundamentals, crystallized by January 2023, when it became clear that the war had entered the attrition phase, as indicated by a flat curve in Figure 1 from month 10 (November 2022).

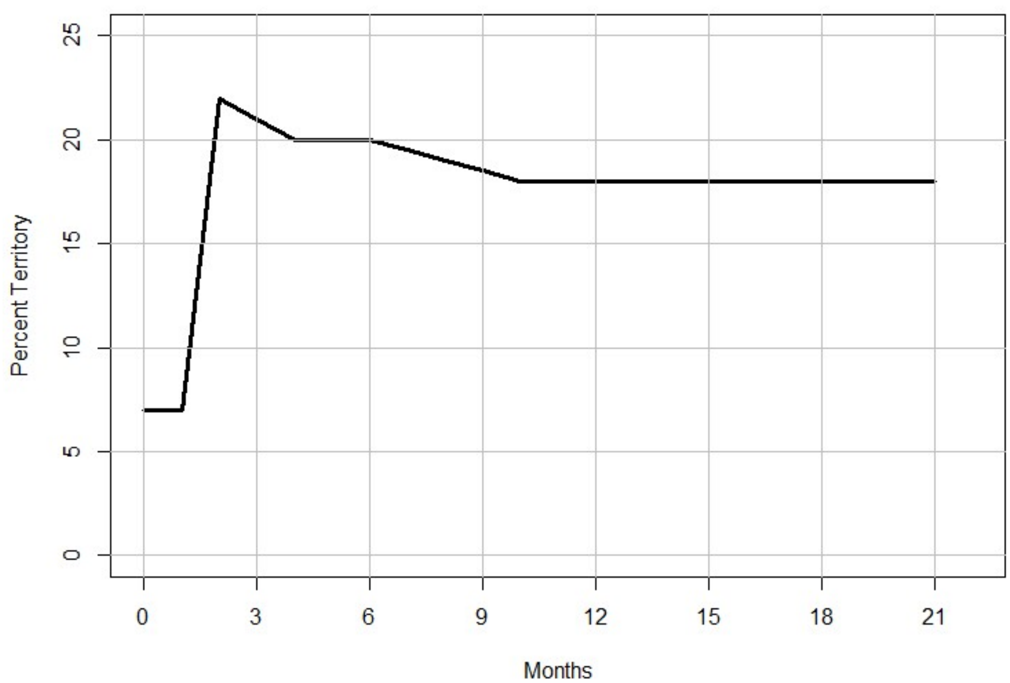

Figure 1. Percent of Ukrainian territory controlled by Russia. X-axis: months since the beginning of the war (1 = February 2022). Data source (O'Hanlon et al. 2023).

The first prediction was based on a comparison of the economic fundamentals characterizing the adversaries, in which Russia (with the GDP of $2 trillion) was contending with the combined economic power of the West ($40 trillion, adding together the GDPs of the USA and EU). In his New York Times column, the Nobelist Paul Krugman argued that this 20-fold differential in economic power provided a decisive advantage to Ukraine (Krugman 2023). "It's true that on the Ukrainian side, Ukrainians are doing all the fighting and dying. But they haven't had to rely on their own military-industrial base," he wrote. Furthermore, "productive capacity—ultimately, economic power—tends to be decisive in a war of attrition. And Russia is just hugely outclassed by that measure. … the brutal slogging match may continue for a very long time. But this is, as I said, largely about math. And the arithmetic, incredibly, seems to favor Ukraine." Here, for specificity, I cite Krugman's column, but this view was expressed by many commentators, and was widely shared among the politicians belonging to the Democratic Party.

This view was not shared by a group of retired military and intelligence professionals, such as Douglas Macgregor, Andrei Martyanov, Ray McGovern, Larry Johnson, and Scott Ritter, who often appear (together and separately) on the podcast *Judging Freedom* hosted by Andrew Napolitano. Again, for specificity, I will cite an opinion written by one of them (Ritter 2023), which was published in January 2023 (as was Krugman's column). Like Krugman's, Ritter's argument is quantitative:

> *With the battlelines currently stabilized, the question of where the war goes from here comes down to basic military math—in short, a causal relationship between two basic equations revolving around burn rates (how quickly losses are sustained) versus replenishment rates (how quickly such losses can be replaced.) The calculus bodes ill for Ukraine.*
>
> *Neither NATO nor the United States appear able to sustain the quantity of weapons that have been delivered to Ukraine, which enabled the successful fall counteroffensives against the Russians.*
>
> *This equipment has largely been destroyed, and despite Ukraine's insistence on its need for more tanks, armored fighting vehicles, artillery and air defense, and while new military aid appears to be forthcoming, it will be late to the battle and in insufficient quantities to have a game-winning impact on the battlefield.*
>
> *Likewise, the casualty rates sustained by Ukraine, which at times reach more than 1,000 men per day, far exceed its ability to mobilize and train replacements.*

Thus, we have two hypotheses, one emphasizing *Economic Power* and the other *Casualty Rates*. Each is based on specific quantitative assumptions, but they predict opposite outcomes for this conflict. Furthermore, I include a third alternative, which differs from the two quantitative ones by positing that military conflicts are essentially unpredictable, because their outcomes are determined by circumstance peculiar to each conflict and random events, including those that we cannot even imagine ahead of time ("unknown unknowns"). Let's refer to the last alternative as the "One Damn Thing After Another" (ODTAA) hypothesis (Turchin 2023).

In the next section I show that the Economic Power, Casualty Rates, and a mild version of the ODTAA hypotheses can be conceptually dealt with within a single mathematical framework.

The Mathematical Framework

The starting point, or perhaps inspiration, for the theory developed here is a mathematical model, independently proposed during World War I by the Russian military officer Mikhail Osipov in 1915 and the English engineer Fredrick Lanchester in 1916 (Turchin 2023: Chapter A1). The core of the Osipov-Lanchester model is the dynamics of casualty rates inflicted by each army shooting projectiles at the enemy. This model is usually applied to a single battle, but I modified it to serve as a model for the course of an entire war. Furthermore, I extend the model by adding equations describing the dynamics of military production by each of the adversaries. Because this model differs in many respects from the Osipov-Lanchester equations, I refer to it as the Attrition War Model (AWM).

The first component that we need to model is productive capacities, P_i, which provide the material basis that is necessary for conducting military operations. Here $i = 1, 2$ indexes the two combatants. The AWM assumes that each combatant starts with a level of production, $P_i(0)$, that initially grows linearly at the rate a_i, but eventually saturates at S_i, the level corresponding to the full mobilization of the economy for warfare purposes (and, therefore, proportional to the overall GDP of the country). This gives us the following simple differential equation:

$$\frac{dP_i}{dt} = a_i\left(1 - \frac{P_i}{S_i}\right)$$

Next, we add a component tracking stocks of warfare materiel, Q_i:

$$\frac{dQ_i}{dt} = b_i P_i - g_i Q_i$$

The assumption here is that Q is augmented by production and depleted at a rate proportional to the overall stock (as the stocks of ammunition, for example, are depleted, each army reduces expenditures in order not to run out at some critical point in future).

Casualties inflicted on the enemy are proportional to the rate of war materiel expenditure, $g_i Q_i$:

$$\frac{dC_j}{dt} = d_i g_i Q_i$$

Note that the index associated with C is j (if $i = 1$, $j = 2$, and vice versa). Here d_i is the efficiency rate (number of casualties per unit of materiel expended). Differences between the two armies in skill, morale, or quality of armaments may be taken into account by assuming different values for their d.

Finally, the dynamics of army sizes, N, are governed by the following equation:

$$\frac{dN_j}{dt} = h_j\left(1 - \frac{N_j + C_j}{M_j}\right) - \ d_i g_i Q_i$$

where the second term represents casualties inflicted by the enemy, while the first term represents recruitment from the population. Initially, recruitment occurs at a maximum rate, h_i, but as the recruitment pool, M, is exhausted, the overall recruitment rate declines, and eventually approaches 0. M is related to the total population of each country.

Eight equations (four variables, P, Q, C, and N for the two adversaries, $i = 1, 2$, see Table 1) constitute the deterministic part of the model. Because the AWM includes both production and attrition components, depending on its parameters and initial conditions it can produce both outcomes, predicted by the Economic Power and Casualties Rate hypotheses (examples are discussed in section *An Interim Assessment*). One way in which we can add the third hypothesis, ODTAA, is by including an explicit random component in the model. I accomplish this by making model parameters (*a*, *b*, *g* etc.) random functions. At each time step, the basic values of the parameter are multiplied by (1 + ε), where ε is a Gaussian-distributed random variable with mean 0 and variance σ^2. This approach, however, allows us to handle only a mild, quantitative version of the ODTAA hypothesis. Major qualitative changes that cannot be foreseen ahead of time ("unknown unknowns") may completely change the nature of the conflict and make all calculations irrelevant.

This Attrition Warfare Model makes a number of simplifying assumptions. First, it folds all types of war materiel—guns, tanks, airplanes, as well as munitions—into a single category. Analysts, such as Trevor Dupuy (1987) have developed elaborate schemes for combining different kinds of armaments into a single measure of military power. In the *Interim Assessment* (see below) I will follow a simplified approach by focusing on the most important casualty-producing materiel in this conflict. After the end of the conflict, using more detailed data, it will be possible to explore how much this simplifying assumption affects the accuracy of forecasts.

Second, the model ignores the difference between offense and defense. This is a reasonable assumption for a relatively static nature of wars of attrition but may need to be modified for more dynamic conflicts. Third, the model lacks a spatial component and possible logistics issues. This may be particularly important when one of the adversaries needs to move war materiel across long distances.

Such simplifying assumptions can be addressed by explicitly modeling additional processes (e.g., movement of war materiel in space). But there is a virtue in keeping the model simple. Ultimately, whether we need to elaborate, or include additional mechanisms into the model will become clear when the model predictions are confronted with data.

Processes	**State Variables**	**Parameters**	**Parameter Explanation**
Military Materiel Production	*P*	*a* *S*	Initial rate of growth of *P* Maximum level of production (*P*)
Materiel Stocks	*Q*	*b* *g*	Conversion rate of production to materiel Depletion rate of materiel
Cumulative Casualties	*C*	*d*	Conversion rate of expended materiel to casualties
Army Size	*N*	*h* *M*	Initial recruitment rate Recruitment pool
Random Forces	ε	σ^2	Variance

Table 1. Model variables and parameters. Subscripts (*i* = 1, 2 indicating assignment to one of the two opponents) are suppressed for clarity.

Analysis Plan

Input and Output Variables

The Attrition Warfare Model (AWM) takes as inputs the values of parameters (Table 1) and estimates of initial conditions. The main drivers of dynamics in the model are Military Material Production, accumulated Material Stocks, and the casualty rates that are inflicted on the enemy by expending military materiel. The outputs that the model predicts are Cumulative Casualties and Army Sizes.

One empirical test of the model is, then, to compare the dynamics of *C* and *N*, predicted by the AWM, to the numbers observed (after the war ends). However, we also need to translate these numbers into a direct measure of war outcome. Qualitatively, the war can end in either victory for one of the sides, or in a stalemate. There is also a fourth possibility, suggested by the strong version of the ODTAA hypothesis: some unforeseen major event will make the model irrelevant.

As a quantitative measure of outcome, which includes all qualitative end points, we can use the proportion of territory of Ukraine in 2013 (that is, before the start of the civil war and the annexation of Crimea by Russia in 2014) held by Russia at the end of war. Thus, the victory by Ukraine translates into driving this proportion to 0, as the oft-stated Ukrainian goal has been recovery of all territories, including Crimea.

Russia has been less explicit about its territorial goals, but several highly placed officials indicated that the aim is to annex all provinces with predominantly Russian population (as indicated, for example, by the proportion of population voting for Yanukovych in the 2010 presidential elections). In addition to the five regions already annexed by Russia, this would also include Odessa, Nikolaev, Dnepropetrovsk, and Kharkov.

A stalemate corresponds to the current (as of November 2023) situation, with Russia holding about 18 percent of Ukrainian territory (Figure 1).

How can we translate the outputs of the AWM (*C*, *N*) into the war outcome, measured by territorial gain/loss? One possibility is when the army size of one of the sides in the conflict falls to a very low level, so that it is no more possible for it to hold the front. However, most wars of attrition end before the army of the losing side is completely destroyed. Instead, the surrender occurs when the defeated country experiences unsupportable warfare losses in military personnel. In other words, when *C* exceeds a certain threshold level, C_{end}.

To gain a quantitative estimate of this level of casualties, I turn to the historical record. I constructed a sample of relevant wars using the data gathered by the Correlates of War project by selecting all wars in this sample that fitted the following criteria: (1) the conflict lasted more than 2 years (thus, ensuring that the war entered an attrition phase), and (2) the defeated side incurred more than 100,000 casualties (as an indicator that the defeated side was determined to win the war, but had to give up under the weight of unsupportable casualties). Table 2 provides the details.

War	**Years**	**State**	**Battle Deaths** (thousands)	**Population** (million)	C_{end} (percent)
World War II (Europe)	1939–45	Germany	3,500	79.0	4.43
Iran-Iraq War*	1980–88	Iraq	500	13.6	3.68
Korean War*	1950–53	North Korea	316	10.5	3.02
American Civil War	1861–65	Confederate States	258	8.7	2.97
World War I	1914–18	Germany	1,773	67.0	2.65
World War II (Pacific)	1941–45	Japan	1,740	73.0	2.38
Iran-Iraq War*	1980–88	Iran	750	38.5	1.95
Vietnam War	1965–75	South Vietnam	254	37.5	0.68
Korean War*	1950–53	South Korea	113	19.0	0.60

Table 2. Wars of attrition in the Correlates of War sample. State: the defeated side or both sides in wars that ended in a stalemate (indicated by *).

Two of the wars in the sample, Iran-Iraq (1980–88) and Korean (1950–53), ended in a stalemate, in which both sides were exhausted, and thus Table 2 lists both sides.

Somewhat surprisingly, the values of C_{end}, calculated as the percentage of population (at start of war) that was killed in battles, cluster reasonably tightly, with most values in the range of 2–3% (Table 2). The median is 2.65 percent. As the population of Ukraine (the part that is not controlled by Russia) is roughly 28 million, the estimate of C_{end} for it is 741,000. Clearly, there is a lot of uncertainty associated with this estimate, so it is safer to use the range of [500,000–1,000,000]. The corresponding value for Russia is 3–4 million.

In summary, the AWM predicts defeat of the state whose casualties level reaches the C_{end} level much earlier than the same happens to its adversary.

Estimating Model Inputs: An Interim Assessment

Currently (as of November 2023) it is not possible to estimate model inputs (parameter values and initial conditions) with any precision. The officials of both Ukraine and Russia are either not reporting key indicators, or quote numbers that wildly disagree. This is as expected while the conflict is in its active phase. Unfortunately, assessments from non-governmental OSINT (open-source intelligence) organizations also differ dramatically among themselves, often reflecting whether they adopt pro-Ukraine or pro-Russia positions. Thus, a proper assessment of model predictions will be possible only after the war's end or, even more likely, several years after that. The purpose of this preregistration, as was stated in the Summary, is to lay out the process of testing the model in a transparent manner that will not allow any leeway to the analyst later, when reasonably accurate estimates become available, and when the ultimate outcome is known.

In July 2023 I conducted a preliminary analysis using what data were available at the time (see War in Ukraine III: an Interim Assessment). I duplicate it here. This analysis was based on decidedly non-authoritative sources (biased in favor of one of the warring sides; some anonymous). The result, thus, is not definitive but, rather, indicative. The purpose of reporting it here is to make transparent, as much as possible, the procedures of testing model predictions when authoritative data become available.

Before the Fall 2023 over 80 percent of casualties in the Ukrainian conflict had been inflicted by artillery. More recently, this preponderance of artillery has been waning due to the increased role of loitering munitions—"kamikaze drones"—and this will need to be assessed in the analysis of the complete course of the war. Additional sources of casualties include rockets and missiles, mines, aircraft, and small arms used by infantry.

However, for the first 1.5 years of conflict, to a first degree of approximation, we need to know how many shells were fired by each side per unit of time. There is a general agreement by all sides that the Russian forces expended many more munitions than the Ukrainians. Specific numbers might be something like 5,000 shells per day fired by the Ukrainians as compared to 20,000 shells fired by the Russians. These numbers primarily refer to heavy guns, shooting 152 mm ammunition (USSR/Russian standard) or 155 mm shells (NATO standard). The numbers represent averages. For example, Russian ammunition expenditures have varied between 10,000 and 50,000 shells per day, or even more. The overall conclusion is that during this period Russia had roughly a 4:1 advantage in artillery.

According to some estimates, the Russian stock of ammunitions at the beginning of conflict was 10–20 million of shells and Russian productive capacity was roughly 150k shells per month. This production rate was subsequently increased by a factor of 3–4. Meanwhile, the combined productive capacities of the US and EU started from 30k shells per month and increased six-fold to 180k.

To illustrate how these data can be used in estimating model parameters, let's assume that the initial stock of shells on the Russian side, $Q_2(0)$ = 10 million. Because an average daily expenditure, g_2Q_2 = 20,000 shells per day, this implies that g_2 = 20,000/10,000,000 = 0.002 day^{-1}. Assuming that the Ukrainian depletion rate is the same as Russian, implies $Q_1(0)$ = 2.5 million shells at the start of the conflict. Furthermore, $b_1 = b_2 = 1$, because I expressed both *P* and *Q* in the same units (artillery shells).

To estimate *d*, the number of casualties per expended shells, I use the study by the consortium of Meduza, Mediazona, and BBC (Meduza 2023). They estimated that 15 months after the beginning of the war, total Russian military casualties (including Donbass militia) were nearly 75,000. Thus, d_1 = 75,000/(5,000x450) = 1/30. I set $d_2 = d_1$. Parameters governing recruitment are estimated analogously. Once again, the specific values are listed here as an illustration of the approach; all estimates will change once authoritative data sources are used after the end of war.

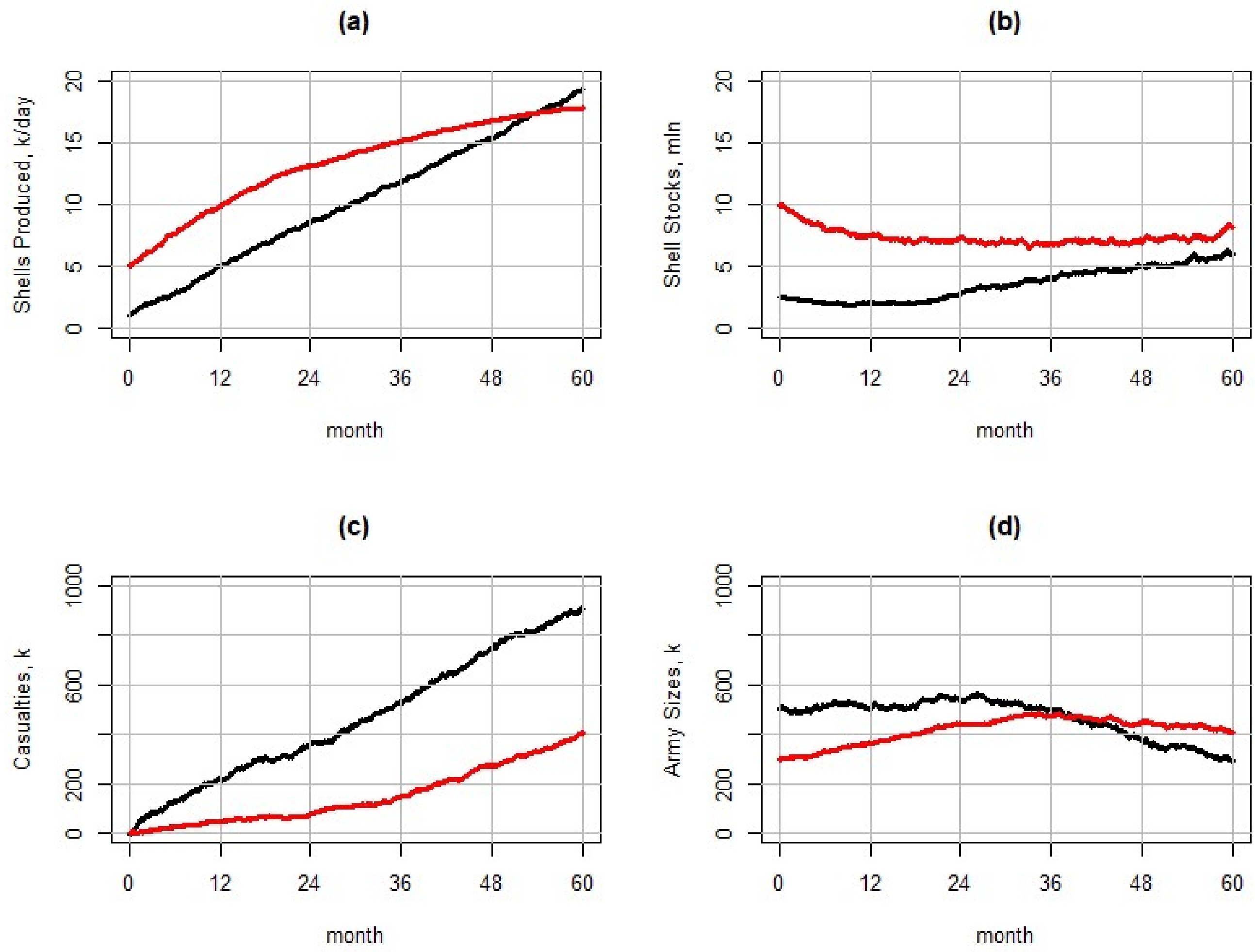

Figure 2. An example of dynamics predicted by the AWM. Variables: (a) *P*, (b) *Q*, (c) *C*, (d) *N*. Black and red colors are for Ukraine and Russia, respectively.

Running these numbers forward produces dynamics depicted in Figure 2. According to this projection, the most important result is that cumulative Ukrainian casualties curve grows faster than that of Russia. Because the model includes a mild version of the ODTAA, different realizations result in somewhat different trajectories (Figure 3). However, assuming that the estimated values of the model parameters are in the right ballpark, the final outcome appears not to be in doubt: the Ukrainian casualties become unsustainable well before the Russian casualties.

We can also use the AWM to investigate counterfactuals, such as, what would it take to reverse the outcome? After all, the Economic Power hypothesis is also encoded in the AWM equations. Numerical investigation of trajectories predicted by the model for different sets of parameter values suggests that it is not easy to overcome the two major advantages enjoyed by Russia over Ukraine: much larger population (and, therefore, higher C_{end} threshold) and the (initial) ammunition advantage. For most parameters, Ukrainian C rapidly grows towards C_{end}, leaving little chance for reversing the impending Ukrainian defeat.

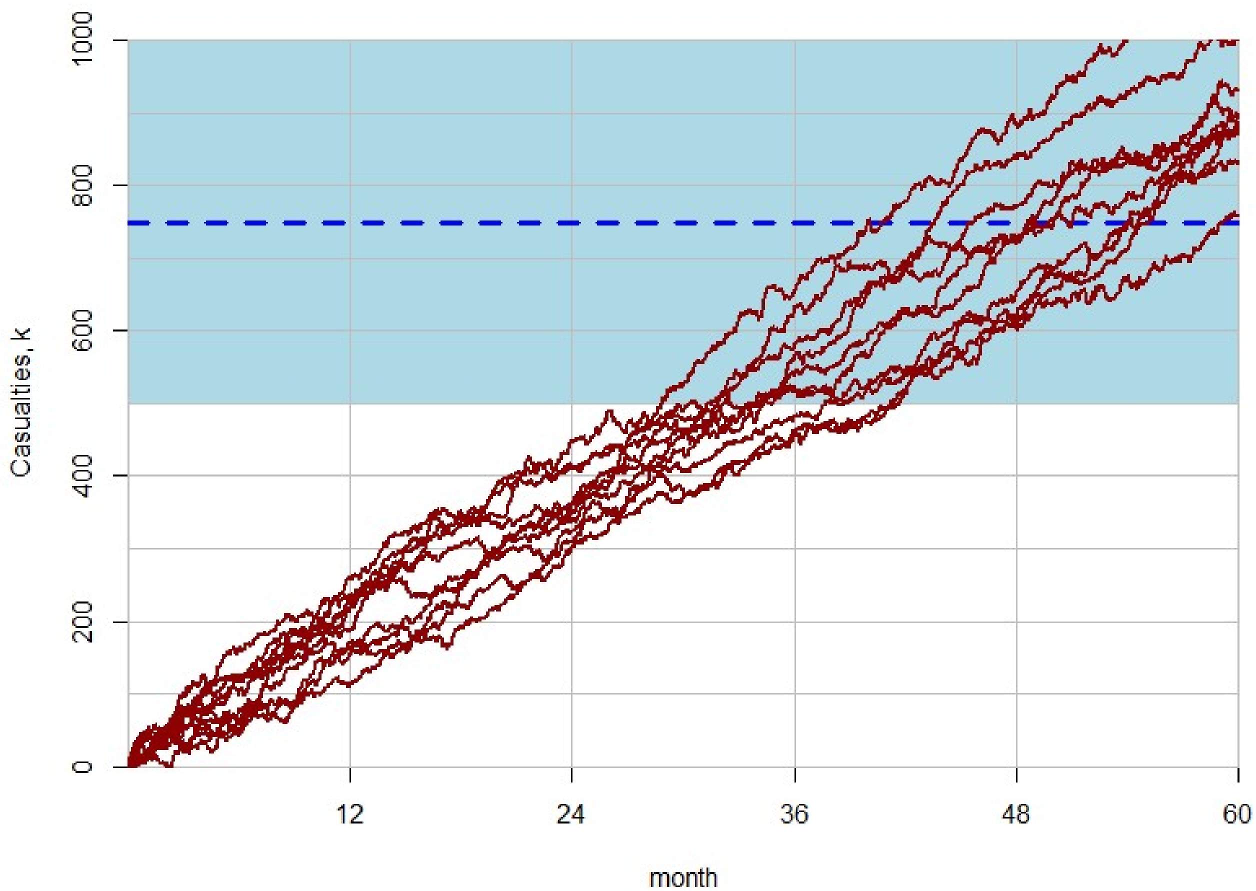

Figure 3. Ten realizations generated by the AWM for the dynamics of C_1 (Ukrainian casualties), assuming $\sigma^2 = 2$. The blue band represents the estimated C_{end} = [500k, 1000k].

Essentially, reversing the predicted outcome requires that the West had succeeded in strangling Russian economy with sanctions *and* had increased its materiel production much more rapidly than it actually did. Figure 4 shows the dynamics predicted by the model for this scenario. Specifically, this version assumes that Russian materiel production, instead of increasing (as in Figure 2), declined from the initial 5,000 shells per day to only 1,000 shells per day. Additionally, the US and EU increased their munitions production at twice the rate, compared to that in Figure 2. The rest of the changes flows from these two assumptions.

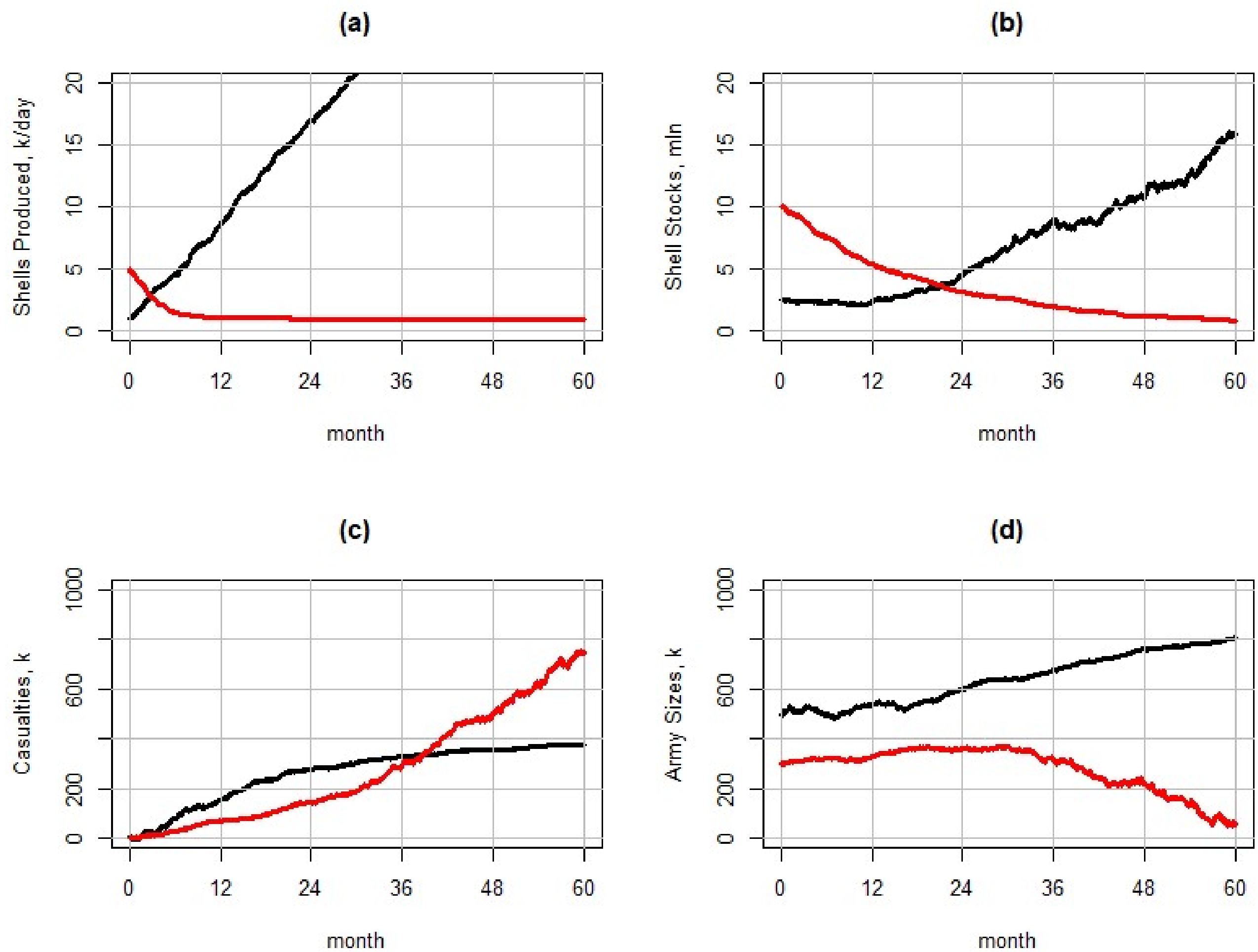

Figure 4. An example of dynamics predicted by the AWM for the Economic Power scenario. Variables: (a) *P*, (b) *Q*, (c) *C*, (d) *N*. Black and red colors are for Ukraine and Russia, respectively.

Model Script

The R Script, which was used in generating AWM predictions (Figures 2 and 4), is posted in open access as an OSF Public Project associated with this pre-registration document. Users can edit it by changing parameter values and initital conditions, and then determine how the resulting dynamics are affected by these choices.

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